

Surfaces of revolution satisfying $\Delta^{III}\mathbf{x} = A\mathbf{x}$

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Abstract

We consider surfaces of revolution in the three-dimensional Euclidean space which are of coordinate finite type with respect to the third fundamental form III , i.e., their position vector \mathbf{x} satisfies the relation $\Delta^{III}\mathbf{x} = A\mathbf{x}$, where A is a square matrix of order 3. We show that a surface of revolution satisfying the preceding relation is a catenoid or part of a sphere.

Key Words: Surfaces in the Euclidean space, surfaces of coordinate finite type, Beltrami operator

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1 Introduction

Let $\mathbf{x} = \mathbf{x}(u^1, u^2)$ be a regular parametric representation of a surface S in the Euclidean space \mathbb{R}^3 which does not contain parabolic points. For two sufficient differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ the first Beltrami operator with respect to the third fundamental form $III = e_{ij}du^i du^j$ of S is defined by

$$\nabla^{III}(f, g) = e^{ij} f_{/i} g_{/j},$$

where $f_{/i} := \frac{\partial f}{\partial u^i}$ and e^{ij} denote the components of the inverse tensor of e_{ij} . The second Beltrami differential operator with respect to III is defined by ¹

$$\Delta^{III}f = \frac{-1}{\sqrt{e}} (\sqrt{e} e^{ij} f_{/i})_{/j} \quad (1)$$

($e := \det(e_{ij})$). In [5] we showed the relation

$$\Delta^{III}\mathbf{x} = \nabla^{III}\left(\frac{2H}{K}, \mathbf{n}\right) - \frac{2H}{K}\mathbf{n}, \quad (2)$$

¹with sign convention such that $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$

where \mathbf{n} is the unit normal vectorfield, H the mean curvature and K the Gaussian curvature of S . Moreover we proved that a surface satisfying the condition

$$\Delta^{III}\mathbf{x} = \lambda\mathbf{x}, \quad \lambda \in \mathbb{R},$$

i.e., a surface $S : \mathbf{x} = \mathbf{x}(u^1, u^2)$ for which all coordinate functions are eigenfunctions of Δ^{III} with the same eigenvalue λ , is part of a sphere ($\lambda = 2$) or a minimal surface ($\lambda = 0$). Using terms of B.-Y. Chen's theory of finite type surfaces [1] the above result can be expressed as follows: *A surface S in \mathbb{R}^3 is of III-type 1 (or of null III-type 1) if and only if S is part of a sphere (or a minimal surface).*

In general a surface S is said to be *of finite type* with respect to the fundamental form III or, briefly, *of finite III-type*, if the position vector \mathbf{x} of S can be written as a finite sum of nonconstant eigenvectors of the operator Δ^{III} , that is if

$$\mathbf{x} = \mathbf{c} + \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_m, \quad \Delta^{III}\mathbf{x}_i = \lambda_i\mathbf{x}_i, \quad i = 1, \dots, m, \quad (3)$$

where \mathbf{c} is a constant vector and $\lambda_1, \dots, \lambda_m$ are eigenvalues of Δ^{III} . When there are exactly k nonconstant eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ appearing in (3) which all belong to different eigenvalues $\lambda_1, \dots, \lambda_k$, then S is said to be of *III-type k* ; when $\lambda_i = 0$ for some $i = 1, \dots, k$, then S is said to be of *null III-type k* .

The only known surfaces of finite III-type are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces (which are actually of null III-type 2, see [5]).

In this paper we want to determine the connected surfaces of revolution S in \mathbb{R}^3 which are *of coordinate finite III-type*, i.e., their position vectorfield $\mathbf{x}(u^1, u^2)$ satisfies the condition

$$\Delta^{III}\mathbf{x} = A\mathbf{x}, \quad A \in M(3, 3), \quad (4)$$

where $M(m, n)$ denotes the set of all matrices of the type (m, n) .

Coordinate finite type surfaces with respect to the first fundamental form I were studied in [2] and [3]. In the last paper O. Garay showed that the only complete surfaces of revolution in \mathbb{R}^3 , whose component functions are eigenfunctions of their Laplacian are the catenoids, the spheres and the circular cylinders, while F. Dillen, J. Pas and L. Verstraelen proved in [2] that the only surfaces in \mathbb{R}^3 satisfying

$$\Delta^I\mathbf{x} = A\mathbf{x} + B, \quad A \in M(3, 3), \quad B \in M(3, 1),$$

are the minimal surfaces, the spheres and the circular cylinders.

Our main result is the following

Proposition 1 *A surface of revolution S satisfies (4) if and only if S is a catenoid or part of a sphere.*

We first show that the mentioned surfaces indeed satisfy the condition (4).

- A. On a catenoid the mean curvature vanishes, so, by virtue of (2), $\Delta^{III}\mathbf{x} = 0$. Therefore a catenoid satisfies (4), where A is the null matrix in $M(3, 3)$.
- B. Let S be part of a sphere of radius r centered at the origin. Then

$$H = \frac{1}{r}, \quad K = \frac{1}{r^2}, \quad \mathbf{n} = -\frac{1}{r}\mathbf{x}.$$

So, by (2), it is $\Delta^{III}\mathbf{x} = 2\mathbf{x}$. Therefore S satisfies (4) with $A = 2I_3$, where I_3 is the identity matrix in $M(3, 3)$.

2 Proof of the main theorem

Let C be the profile curve of a surface of revolution S of the differentiation class C^3 . We suppose that (a) C lies on the (x_1, x_3) -plane, (b) the axis of revolution of S is the x_3 -axis and (c) C is parametrized by its arclength s . Then C admits the parametric representation

$$\mathbf{r}(s) = (f(s), 0, g(s)), \quad s \in J$$

($J \subset \mathbb{R}$ open interval), where $f(s), g(s) \in C^3(J)$. The position vector of S is given by

$$\mathbf{x}(s, \theta) = (f(s) \cos \theta, f(s) \sin \theta, g(s)), \quad s \in J, \quad \theta \in [0, 2\pi).$$

Putting $f'(s) := \frac{df(s)}{ds}$ we have because of (c)

$$f'^2 + g'^2 = 1 \quad \forall s \in J. \quad (5)$$

Furthermore it is $f' \cdot g' \neq 0$, because otherwise $f = \text{const.}$ or $g = \text{const.}$ and S would be a circular cylinder or part of a plane, respectively. Hence S would consist only of parabolic points, which has been excluded. In view of (5) we can put

$$f' = \cos \varphi, \quad g' = \sin \varphi, \quad (6)$$

where φ is a function of s . Then the unit normal vector of S is given by

$$\mathbf{n} = (-\sin \varphi \cos \theta, -\sin \varphi \sin \theta, \cos \varphi).$$

The components h_{ij} and e_{ij} of the second and the third fundamental tensors in (local) coordinates are the following

$$\begin{aligned} h_{11} &= \varphi', & h_{12} &= 0, & h_{22} &= f \sin \varphi, \\ e_{11} &= \varphi'^2, & e_{12} &= 0, & e_{22} &= \sin^2 \varphi, \end{aligned} \quad (7)$$

hence [4]

$$\frac{2H}{K} = h_{ij}e^{ij} = \frac{1}{\varphi} + \frac{f}{\sin \varphi}. \quad (8)$$

From (1) and (7) we find for a sufficient differentiable function $u = u(s, \theta)$ defined on $J \times [2\pi, 0)$

$$\Delta^{III} u = -\frac{u''}{\varphi'^2} + \left(\frac{\varphi''}{\varphi'^2} - \frac{\cos \varphi}{\sin \varphi} \right) \frac{u'}{\varphi'} - \frac{u_{/\theta\theta}}{\sin^2 \varphi}. \quad (9)$$

Considering the following functions of s

$$P_1 = R \sin \varphi - \frac{\cos \varphi}{\varphi'} R', \quad P_2 = -R \cos \varphi - \frac{\sin \varphi}{\varphi'} R', \quad (10)$$

where we have put for simplicity $R := \frac{2H}{K}$, and applying (9) on the coordinate functions x_i , $i = 1, 2, 3$, of the position vector \mathbf{x} we find

$$\Delta^{III} x_1 = P_1 \cos \theta, \quad \Delta^{III} x_2 = P_1 \sin \theta, \quad \Delta^{III} x_3 = P_2. \quad (11)$$

So we have:

(a) *The coordinate functions x_1, x_2 are both eigenfunctions of Δ^{III} belonging to the same eigenvalue if and only if for some real constant λ holds*

$$\lambda f = R \sin \varphi - \frac{\cos \varphi}{\varphi'} R'.$$

(b) *The coordinate function x_3 is an eigenfunction of Δ^{III} if and only if for some real constant μ holds*

$$\mu g = -R \cos \varphi - \frac{\sin \varphi}{\varphi'} R'.$$

We denote by a_{ij} , $i, j = 1, 2, 3$, the entries of the matrix A . By using (11) condition (4) is found to be equivalent to the following system

$$\begin{cases} P_1 \cos \theta = a_{11} f \cos \theta + a_{12} f \sin \theta + a_{13} g \\ P_1 \sin \theta = a_{21} f \cos \theta + a_{22} f \sin \theta + a_{23} g \\ P_2 = a_{31} f \cos \theta + a_{32} f \sin \theta + a_{33} g \end{cases} \quad (12)$$

Since $\sin \theta, \cos \theta$ and 1 are linearly independent functions of θ , we obtain from (12₃) $a_{31} = a_{32} = 0$. On differentiating (12₁) and (12₂) twice with respect to θ we have

$$\begin{cases} P_1 \cos \theta = a_{11} f \cos \theta + a_{12} f \sin \theta \\ P_1 \sin \theta = a_{21} f \cos \theta + a_{22} f \sin \theta \end{cases}.$$

Thus $a_{13}g = a_{23}g = 0$, so that a_{13} and a_{23} vanish. The system (12) is equivalent to the following

$$\begin{cases} (P_1 - a_{11}f) \cos \theta - a_{12}f \sin \theta = 0 \\ (P_1 - a_{22}f) \sin \theta - a_{21}f \cos \theta = 0 \\ P_2 - a_{33}g = 0 \end{cases}.$$

But $\sin \theta$ and $\cos \theta$ are linearly independent functions of θ , so we finally obtain $a_{12} = a_{21} = 0$, $a_{11} = a_{22}$ and $P_1 = a_{11}f$. Putting $a_{11} = a_{22} = \lambda$ and $a_{33} = \mu$ we see that the system (12) reduces now to the following equations

$$P_1 = \lambda f, \quad P_2 = \mu g. \quad (13)$$

On account of (10) and (13) we are left with the system

$$\begin{cases} R = \lambda f \sin \varphi - \mu g \cos \varphi \\ R' = -\varphi'(\lambda f \cos \varphi + \mu g \sin \varphi) \end{cases} \quad (14)$$

On differentiating (14₁) with respect to s we find, by virtue of (6),

$$R' = \frac{\lambda - \mu}{2} \sin \varphi \cos \varphi. \quad (15)$$

We distinguish the following cases:

Case I. Let $\lambda = \mu$.

Then (15) reduces to $R' = 0$.

Subcase Ia. Let $\lambda = \mu = 0$. From (14₁) we obtain $R = 0$, i.e., $H = 0$. Consequently S , being a minimal surface of revolution, is a catenoid.

Subcase Ib. Let $\lambda = \mu \neq 0$.

Then from (6), (14₂) and $R' = 0$ we have $f \cdot f' + g \cdot g' = 0$, i.e., $(f^2 + g^2)' = 0$. Therefore $f^2 + g^2 = \text{const.}$ and S is obviously part of a sphere.

Case II. Let $\lambda \neq \mu$. From (14₂), (15) we find firstly

$$\frac{1}{\varphi'} = \frac{2(\lambda f \cos \varphi + \mu g \sin \varphi)}{(\mu - \lambda) \sin \varphi \cos \varphi}. \quad (16)$$

From this and (8) we obtain

$$R = \frac{\lambda + \mu}{(\mu - \lambda) \sin \varphi} f + \frac{2\mu}{(\mu - \lambda) \cos \varphi} g.$$

Hence, by virtue of (14₁),

$$af + bg = 0, \quad (17)$$

where

$$a = \lambda \sin \varphi + \frac{\lambda + \mu}{(\lambda - \mu) \sin \varphi}, \quad b = \frac{2\mu}{(\lambda - \mu) \cos \varphi} - \mu \cos \varphi. \quad (18)$$

We note that $\mu \neq 0$, since for $\mu = 0$ we have

$$a = \frac{\lambda \sin^2 \varphi + 1}{\sin \varphi}, \quad b = 0,$$

and relation (17) becomes

$$\frac{\lambda \sin^2 \varphi + 1}{\sin \varphi} f = 0,$$

whence it follows $\lambda \sin^2 \varphi + 1 = 0$, a contradiction.

On differentiating (17) with respect to s and taking into account (16) we obtain

$$a_1 \frac{f}{\sin \varphi} + b_1 \frac{g}{\cos \varphi} = 0, \quad (19)$$

where

$$a_1 = \lambda(\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\lambda\mu - \lambda^2 + 3\lambda + \mu) \sin^2 \varphi - (\lambda + \mu)(3\lambda - \mu), \quad (20)$$

$$b_1 = \mu \left[(\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\mu - \lambda + 4) \sin^2 \varphi - 2(\lambda + \mu) \right]. \quad (21)$$

By eliminating now the functions f and g from (17) and (19) and taking into account (18), (20) and (21) we find

$$\lambda(\lambda - \mu)^2 \sin^4 \varphi + (\lambda - \mu)(\lambda\mu - \lambda^2 + 5\lambda + \mu - 2) \sin^2 \varphi + (\lambda + \mu)(\mu - 3\lambda + 4) = 0.$$

Consequently

$$\lambda(\lambda - \mu)^2 = 0, \quad (\lambda - \mu)(\lambda\mu - \lambda^2 + 5\lambda + \mu - 2) = 0, \quad (\lambda + \mu)(\mu - 3\lambda + 4) = 0.$$

From the first equation we have $\lambda = 0$. Then, the other two become as follows

$$\mu - 2 = 0, \quad \mu + 4 = 0,$$

which is a contradiction.

So the proof of the theorem is completed.

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